

On the Dynamics of Induced Maps on the Space of Probability Measures

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Abstract

For the generic continuous map and for the generic homeomorphism of the Cantor space, we study the dynamics of the induced map on the space of probability measures, with emphasis on the notions of Li-Yorke chaos, topological entropy, equicontinuity, chain continuity, chain mixing, shadowing and recurrence. We also establish some results concerning induced maps that hold on arbitrary compact metric spaces.¹

1 Introduction

Let M be a compact metric space with metric d and let \mathcal{B}_M be the set of all Borel subsets of M . We denote by $\mathcal{C}(M)$ (resp. $\mathcal{H}(M)$) the space of all continuous maps from M into M (resp. of all homeomorphisms from M onto M) endowed with the metric

$$\tilde{d}(f, g) := \max_{x \in M} d(f(x), g(x)).$$

Moreover, we denote by $\mathcal{K}(M)$ the hyperspace of all nonempty closed subsets of M endowed with the Hausdorff metric

$$d_H(X, Y) := \max \left\{ \max_{x \in X} d(x, Y), \max_{y \in Y} d(y, X) \right\},$$

and by $\mathcal{M}(M)$ the space of all Borel probability measures on M endowed with the Prohorov metric

$$d_P(\mu, \nu) := \inf \{ \delta > 0 : \mu(X) \leq \nu(X^\delta) + \delta \text{ and } \nu(X) \leq \mu(X^\delta) + \delta \text{ for all } X \in \mathcal{B}_M \},$$

where $X^\delta := \{x \in M : d(x, X) < \delta\}$ is the δ -neighborhood of X ($X \subset M$). The Prohorov metric induces the usual weak topology for measures. It is well known that both $\mathcal{K}(M)$ and $\mathcal{M}(M)$ are compact metric spaces. Moreover,

$$d_P(\mu, \nu) = \inf \{ \delta > 0 : \mu(X) \leq \nu(X^\delta) + \delta \text{ for all } X \in \mathcal{B}_M \}$$

([12], Page 72). Given $f \in \mathcal{C}(M)$, the induced maps $\bar{f} : \mathcal{K}(M) \rightarrow \mathcal{K}(M)$ and $\tilde{f} : \mathcal{M}(M) \rightarrow \mathcal{M}(M)$ are the continuous maps given by

$$\bar{f}(X) := f(X) \quad (X \in \mathcal{K}(M))$$

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and

$$(\tilde{f}(\mu))(X) := \mu(f^{-1}(X)) \quad (\mu \in \mathcal{M}(M), X \in \mathcal{B}_M).$$

If f is a homeomorphism, then so are \bar{f} and \tilde{f} . We refer the reader to the books [25] and [12] for a study of the spaces $\mathcal{K}(M)$ and $\mathcal{M}(M)$, respectively.

Given a Baire space Z , to say that “the generic element of Z has a certain property P ” means that the set of all elements of Z that do not satisfy property P is of the first category in Z . The word “typical” is sometimes used instead of the word “generic”.

A systematic study of the dynamics of the induced maps \bar{f} and \tilde{f} was initiated by Bauer and Sigmund [6] and has been developed by several authors; see [1, 21, 23, 32], for instance. On the other hand, the study of generic dynamics is a classical topic in the area of dynamical systems. In the context of topological dynamics, such a study has been developed by several authors during the last forty years. We refer the reader to [3, 5, 8, 9, 10, 29], where further references can be found.

In [11] the authors combined both topics and developed a detailed study of the dynamics of the induced map \bar{f} in the case f is the generic continuous map or the generic homeomorphism of the Cantor space. Our main goal in the present paper is to develop such a detailed study for the induced map \tilde{f} . It turns out that in many aspects the dynamics of the induced map \tilde{f} is completely different from the dynamics of the induced map \bar{f} . For instance, for the generic homeomorphism f of the Cantor space, it was proved in [11] that the induced map \bar{f} is uniformly distributionally chaotic, has infinite topological entropy, has the shadowing property and is chain continuous at every point of a dense open set, but we will see that the induced map \tilde{f} has no Li-Yorke pair, has zero topological entropy, does not have the shadowing property and is chain continuous at no point at all.

We also consider the problem of the density of the set of periodic points in the set of nonwandering points. The C^1 *closing lemma* and the associated C^1 *general density theorem* are fundamental results in the theory of smooth dynamical systems due to Pugh [30]. The former says that if x is a nonwandering point of a diffeomorphism f on a compact smooth manifold M , then in any C^1 neighborhood of f there is a diffeomorphism g for which x is a periodic point. The latter says that C^1 generically the set of periodic points of a diffeomorphism is dense in the set of its nonwandering points. The corresponding C^0 *general density theorem* for homeomorphisms on a compact smooth manifold M was announced by Palis, Pugh, Shub and Sullivan [28], but a flaw in their argument was later described by Coven, Madden and Nitecki [18], who proposed a different argument. However, it was pointed out by Pilyugin [29] that the argument in [18] only works in the C^0 closure of the set of diffeomorphisms in the set of homeomorphisms on M . By Munkres [27] and Whitehead [33], this C^0 closure is equal to the set of all homeomorphisms on M whenever $\dim M \leq 3$, but Munkres [27] also showed that this is not the case if $\dim M > 3$. A proof of the full C^0 general density theorem was finally given by Hurley [24], who also observed how to adapt the argument to obtain the corresponding result for continuous maps. In the case of the Cantor space the situation is completely different: the generic continuous map and the generic homeomorphism of the Cantor space have no periodic point! This was observed by D’Aniello and Darji [19] and by Akin, Hurley and Kennedy [5], respectively. Nevertheless, surprisingly enough, we will see that for the generic continuous map (resp. the generic homeomorphism) f of the Cantor space, the set of periodic points of the induced map \tilde{f} is dense in the set of its nonwandering points. Moreover, several additional properties of the induced map \tilde{f} related to recurrence will also be established.

Although our main motivation was to study the dynamics of the induced map \tilde{f} in the case f is the generic map of the Cantor space, in doing so we have also established some results that hold on arbitrary compact metric spaces. For instance, we will see that for any homeomorphism f of any compact metric space M , the following properties hold:

- \tilde{f} is chain mixing.
- \tilde{f} has no point of chain continuity (provided M is not a singleton).

Moreover, for any continuous map f of any compact metric space M , the following assertions are equivalent:

- (i) \tilde{f} is chain continuous at some point;
- (ii) \tilde{f} is chain continuous at every point;
- (iii) $\bigcap_{n=1}^{\infty} f^n(M)$ is a singleton.

Such results complement the previous works of Bauer and Sigmund [6], Sigmund [32] and Glasner and Weiss [21] on the dynamics of the induced map \tilde{f} in the context of general compact metric spaces.

2 Preliminaries

Throughout M denotes an arbitrary compact metric space with metric d . For each $x \in M$ and each $r > 0$, $B(x; r) := \{y \in M : d(y, x) < r\}$ is the open ball with center x and radius r .

Our model for the Cantor space is the product space $\{0, 1\}^{\mathbb{N}}$, where $\{0, 1\}$ is endowed with the discrete topology. We consider $\{0, 1\}^{\mathbb{N}}$ endowed with the compatible metric, also denoted by d , given by $d(\sigma, \sigma) := 0$ and $d(\sigma, \tau) := \frac{1}{n}$ where n is the least positive integer such that $\sigma(n) \neq \tau(n)$ ($\sigma, \tau \in \{0, 1\}^{\mathbb{N}}$, $\sigma \neq \tau$).

The main tools used in the present paper for the results concerning the generic continuous map and the generic homeomorphism of the Cantor space are the graph theoretic descriptions of these maps obtained in [10]. For the sake of completeness, let us briefly recall these descriptions.

A *partition* of $\{0, 1\}^{\mathbb{N}}$ is a finite collection \mathcal{P} of pairwise disjoint nonempty clopen sets whose union is $\{0, 1\}^{\mathbb{N}}$, and $\text{mesh}(\mathcal{P})$ is the maximum diameter of the elements of \mathcal{P} . For each $f \in \mathcal{C}(\{0, 1\}^{\mathbb{N}})$ and each partition \mathcal{P} of $\{0, 1\}^{\mathbb{N}}$, we consider the digraph $\text{Gr}(f, \mathcal{P})$ whose vertex set is \mathcal{P} and whose edge set is

$$\{\overrightarrow{ab} : a, b \in \mathcal{P} \text{ and } f(a) \cap b \neq \emptyset\}.$$

A *component* of a digraph G is a largest (in vertices and edges) subgraph H of G such that given any two vertices a, b in H , there are vertices a_1, \dots, a_n in H such that $a_1 = a$, $a_n = b$ and, for each $1 \leq i < n$, $\overrightarrow{a_i a_{i+1}}$ or $\overrightarrow{a_{i+1} a_i}$ is an edge of H .

A digraph ℓ is a *loop of length n* if the vertex set of ℓ is a set $\{v_1, \dots, v_n\}$ with n elements and the edges of ℓ are $\overrightarrow{v_n v_1}$ and $\overrightarrow{v_i v_{i+1}}$ for $1 \leq i < n$.

A digraph B is a *balloon of type (s, t)* if the vertex set of B is the union of two disjoint sets $p = \{v_1, \dots, v_s\}$ and $\ell = \{w_1, \dots, w_t\}$, and the edges of B are the edges of the path

p (i.e., $\overrightarrow{v_i v_{i+1}}$ for $1 \leq i < s$), the edges of the loop formed by ℓ , and $\overrightarrow{v_s w_1}$. We call v_1 the *initial vertex* of B .

A digraph D is a *dumbbell of type (r, s, t)* if the vertex set of D is the union of three disjoint sets $\ell_1 = \{u_1, \dots, u_r\}$, $p = \{v_1, \dots, v_s\}$ and $\ell_2 = \{w_1, \dots, w_t\}$, and the edges of D are the edges of the loops formed by ℓ_1 and ℓ_2 , the edges of the path p , $\overrightarrow{u_1 v_1}$ and $\overrightarrow{v_s w_1}$. If $r = t$ then we say that the dumbbell is *balanced with plate weight r* .

Suppose that $f \in \mathcal{C}(\{0, 1\}^{\mathbb{N}})$, \mathcal{P} is a partition of $\{0, 1\}^{\mathbb{N}}$ and B is a component of $\text{Gr}(f, \mathcal{P})$ which is a balloon. Write

$$B = \{v_1, \dots, v_s\} \cup \{w_1, \dots, w_t\},$$

with usual labeling. We say that the balloon B is *strict relative to f* if $f(v_i) \subsetneq v_{i+1}$ for every $1 \leq i < s$, $f(w_j) \subsetneq w_{j+1}$ for every $1 \leq j < t$, and $f(v_s) \cup f(w_t) \subsetneq w_1$.

Suppose that $h \in \mathcal{H}(\{0, 1\}^{\mathbb{N}})$, \mathcal{P} is a partition of $\{0, 1\}^{\mathbb{N}}$ and D is a component of $\text{Gr}(h, \mathcal{P})$ which is a dumbbell. Write

$$D = \{u_1, \dots, u_r\} \cup \{v_1, \dots, v_s\} \cup \{w_1, \dots, w_t\},$$

with usual labeling. We say that the dumbbell D *contains a left loop of h* (resp. *a right loop of h*) if there is a nonempty clopen subset a of u_1 (resp. of w_1) such that $h^r(a) = a$ (resp. $h^t(a) = a$).

Let us now recall the above-mentioned results from [10]:

Theorem A. *The generic $f \in \mathcal{C}(\{0, 1\}^{\mathbb{N}})$ has the following property:*

(Q) *For every $m \in \mathbb{N}$, there are a partition \mathcal{P} of $\{0, 1\}^{\mathbb{N}}$ of mesh $< 1/m$ and a multiple $q \in \mathbb{N}$ of m such that every component of $\text{Gr}(f, \mathcal{P})$ is a balloon of type $(q!, q!)$ which is strict relative to f .*

Theorem B. *The generic $h \in \mathcal{H}(\{0, 1\}^{\mathbb{N}})$ has the following property:*

(P) *For every $m \in \mathbb{N}$, there are a partition \mathcal{P} of $\{0, 1\}^{\mathbb{N}}$ of mesh $< 1/m$ and a multiple $q \in \mathbb{N}$ of m such that every component of $\text{Gr}(h, \mathcal{P})$ is a balanced dumbbell with plate weight $q!$ that contains both a left and a right loop of h .*

Moreover, it was proved in [10] that any two maps $f, g \in \mathcal{C}(\{0, 1\}^{\mathbb{N}})$ (resp. $f, g \in \mathcal{H}(\{0, 1\}^{\mathbb{N}})$) with property (Q) (resp. property (P)) are topologically conjugate to each other, that is, $f = h^{-1}gh$ for some $h \in \mathcal{H}(\{0, 1\}^{\mathbb{N}})$.

Given a partition \mathcal{P} of $\{0, 1\}^{\mathbb{N}}$, we define

$$\delta(\mathcal{P}) := \min\{d(a, b) : a, b \in \mathcal{P}, a \neq b\} > 0$$

and

$$I_{\mathcal{P}}(X) := \{a \in \mathcal{P} : a \cap X \neq \emptyset\} \quad (X \subset \{0, 1\}^{\mathbb{N}}).$$

For each $z \in M$, $\pi_z \in \mathcal{M}(M)$ denotes the unit mass concentrated at z . Note that

$$d_P(\pi_z, \pi_w) = \min\{d(z, w), 1\}.$$

Moreover, for every $f \in \mathcal{C}(M)$,

$$\tilde{f}(\pi_z) = \pi_{f(z)}.$$

3 Li-Yorke chaos and topological entropy

Let us begin by recalling the notions of Li-Yorke chaos [26] and distributional chaos [31]. Given $f \in \mathcal{C}(M)$, recall that (x, y) is a *Li-Yorke pair* for f if

$$\liminf_{n \rightarrow \infty} d(f^n(x), f^n(y)) = 0 \quad \text{and} \quad \limsup_{n \rightarrow \infty} d(f^n(x), f^n(y)) > 0.$$

The map f is *Li-Yorke chaotic* if there is an uncountable set S (a *scrambled set* for f) such that (x, y) is a Li-Yorke pair for f whenever x and y are distinct points in S . Moreover, (x, y) is a *distributionally ε -chaotic pair* for f ($\varepsilon > 0$) if

$$\overline{\text{dens}}\{n \in \mathbb{N} : d(f^n(x), f^n(y)) \geq \varepsilon\} = 1$$

and

$$\overline{\text{dens}}\{n \in \mathbb{N} : d(f^n(x), f^n(y)) < \delta\} = 1,$$

for all $\delta > 0$, where

$$\overline{\text{dens}}(A) := \limsup_{n \rightarrow \infty} \frac{\text{card}([1, n] \cap A)}{n}$$

is the *upper density* of the subset A of \mathbb{N} . The map f is *uniformly distributionally chaotic* if there is an uncountable set S (a *distributionally ε -scrambled set* for f) such that (x, y) is a distributionally ε -chaotic pair for f whenever x and y are distinct points in S .

It was proved in [11] that for the generic $h \in \mathcal{H}(\{0, 1\}^{\mathbb{N}})$, the induced map \bar{h} is uniformly distributionally chaotic. Surprisingly enough, we shall now see that the induced map \tilde{h} is not even Li-Yorke chaotic. In fact, even the following stronger statement holds.

Theorem 1. *For the generic $h \in \mathcal{H}(\{0, 1\}^{\mathbb{N}})$, \tilde{h} has no Li-Yorke pair.*

Proof. Let $h \in \mathcal{H}(\{0, 1\}^{\mathbb{N}})$ satisfy property (P) of Theorem B. Let (μ, ν) be a pair of elements of $\mathcal{M}(\{0, 1\}^{\mathbb{N}})$ and suppose that

$$\limsup_{n \rightarrow \infty} d_P(\tilde{h}^n(\mu), \tilde{h}^n(\nu)) > 0.$$

Then, we may fix $\varepsilon > 0$ such that

$$d_P(\tilde{h}^n(\mu), \tilde{h}^n(\nu)) > \varepsilon \tag{1}$$

for infinitely many values of n . For each such n , there is a Borel subset Y_n of $\{0, 1\}^{\mathbb{N}}$ such that

$$\mu(h^{-n}(Y_n)) > \nu(h^{-n}((Y_n)^\varepsilon)) + \varepsilon. \tag{2}$$

Let \mathcal{P} be a partition of $\{0, 1\}^{\mathbb{N}}$ of mesh $< \varepsilon$ such that every component of $\text{Gr}(h, \mathcal{P})$ is a balanced dumbbell with plate weight $q \geq 2$. Let

$$D_i := \{u_{i,1}, \dots, u_{i,q}\} \cup \{v_{i,1}, \dots, v_{i,s_i}\} \cup \{w_{i,1}, \dots, w_{i,q}\} \quad (1 \leq i \leq N)$$

be the components (dumbbells) of $\text{Gr}(h, \mathcal{P})$. For each $i \in \{1, \dots, N\}$, we consider the nonempty closed set

$$X_i := F_i \cup h(F_i) \cup \dots \cup h^{q-1}(F_i),$$

where $F_i := \bigcap_{n=0}^{\infty} h^{-nq}(u_{i,1})$. Note that $h(X_i) = X_i$, because $h^q(F_i) = F_i$. Moreover, $(u_{i,1} \cup \dots \cup u_{i,q}) \setminus X_i$ is exactly the set of all $\sigma \in u_{i,1} \cup \dots \cup u_{i,q}$ whose forward trajectory eventually goes to the bar of the dumbbell D_i , that is, $h^r(\sigma) \in v_{i,1}$ for some $r \in \mathbb{N}$.

For each n such that (1) holds, we define $A_n := \bigcup\{a : a \in I_{\mathcal{P}}(Y_n)\} \supset Y_n$. Since $\text{mesh}(\mathcal{P}) < \varepsilon$, $A_n \subset (Y_n)^\varepsilon$. Thus, by (2),

$$\mu(h^{-n}(A_n)) > \nu(h^{-n}(A_n)) + \varepsilon.$$

Since this last inequality holds for infinitely many values of n and there are only finitely many possible A_n 's, we see that there is a set A which is a union of some elements of \mathcal{P} such that

$$\mu(h^{-n}(A)) > \nu(h^{-n}(A)) + \varepsilon \quad (3)$$

for infinitely many values of n .

Since

$$\lim_{k \rightarrow \infty} \varphi\left(\bigcup_{i=1}^N \bigcup_{n=k}^{\infty} h^{-n}(v_{i,1})\right) = 0$$

for every $\varphi \in \mathcal{M}(\{0,1\}^{\mathbb{N}})$, we may fix $k \in \mathbb{N}$ such that

$$\mu(Z) < \varepsilon/3 \quad \text{and} \quad \nu(Z) < \varepsilon/3, \quad (4)$$

where

$$Z := \bigcup_{i=1}^N \bigcup_{n=k}^{\infty} h^{-n}(v_{i,1}).$$

Now, we decompose the set A into three disjoint sets:

$$A = B \cup C \cup D,$$

where

$$\begin{aligned} B &\subset X := \bigcup_{i=1}^N X_i, \\ C &\subset U := \bigcup_{i=1}^N [(u_{i,1} \cup \dots \cup u_{i,q}) \setminus X_i] \cup (v_{i,1} \cup \dots \cup v_{i,s_i}), \\ D &\subset W := \bigcup_{i=1}^N (w_{i,1} \cup \dots \cup w_{i,q}). \end{aligned}$$

Since $h^{-q}(B) = B$, $h^{-q}(D) \cap W = D$ and $h^{-n}(C) \subset h^{-n}(U) \subset Z$ whenever n is big enough, it follows that there exists $n_0 \in \mathbb{N}$ such that the sequence

$$(h^{-n}(A) \setminus Z)_{n \geq n_0}$$

is periodic and its period divides q . By (3), there exists $m_0 \geq n_0$ such that

$$\mu(h^{-m_0}(A)) > \nu(h^{-m_0}(A)) + \varepsilon.$$

Hence, by (4),

$$\mu(h^{-m_0}(A) \setminus Z) > \nu(h^{-m_0}(A) \setminus Z) + \frac{2\varepsilon}{3}.$$

Since the period of the periodic sequence $(h^{-n}(A) \setminus Z)_{n \geq n_0}$ divides q , we obtain

$$\mu(h^{-m_0-nq}(A) \setminus Z) > \nu(h^{-m_0-nq}(A) \setminus Z) + \frac{2\varepsilon}{3} \quad \text{for all } n \in \mathbb{N}_0. \quad (5)$$

Choose $0 < \delta < \min\{\delta(\mathcal{P}), \varepsilon/3\}$. Then, for every $n \in \mathbb{N}_0$,

$$\begin{aligned}\mu(h^{-m_0-nq}(A)) &\geq \mu(h^{-m_0-nq}(A) \setminus Z) \\ &> \nu(h^{-m_0-nq}(A) \setminus Z) + \frac{2\varepsilon}{3} \\ &> \nu(h^{-m_0-nq}(A)) + \frac{\varepsilon}{3} \\ &> \nu(h^{-m_0-nq}(A)) + \delta \\ &= \nu(h^{-m_0-nq}(A^\delta)) + \delta,\end{aligned}$$

where we used (5), (4) and the fact that $A^\delta = A$ (because $\delta < \delta(\mathcal{P})$). Thus,

$$d_P(\tilde{h}^{m_0+nq}(\mu), \tilde{h}^{m_0+nq}(\nu)) \geq \delta \quad \text{for all } n \in \mathbb{N}_0.$$

This implies that

$$\liminf_{n \rightarrow \infty} d_P(\tilde{h}^n(\mu), \tilde{h}^n(\nu)) > 0,$$

and so (μ, ν) is not a Li-Yorke pair for \tilde{h} . □

Let us now recall the notion of topological entropy. Fix $f \in \mathcal{C}(M)$. For each $n \in \mathbb{N}$, consider the equivalent metric d_n on M given by

$$d_n(x, y) := \max_{0 \leq k < n} d(f^k(x), f^k(y)).$$

A subset A of M is (n, ϵ, f) -separated if $d_n(x, y) \geq \epsilon$ for every $x, y \in A$ with $x \neq y$. Let $N(n, \epsilon, f)$ be the maximum cardinality of an (n, ϵ, f) -separated set. The *topological entropy* of f is defined by

$$\text{ent}(f) := \lim_{\epsilon \rightarrow 0^+} \left(\limsup_{n \rightarrow \infty} \frac{1}{n} \log N(n, \epsilon, f) \right).$$

This notion was introduced by Adler, Konheim and McAndrew [2]. Here we are adopting the equivalent definition formulated by Bowen [14] and Dinaburg [20].

Glasner and Weiss [21] discovered the surprising fact that $\text{ent}(h) = 0$ implies $\text{ent}(\tilde{h}) = 0$ ($h \in \mathcal{H}(M)$). Moreover, they proved in [22] that the generic $h \in \mathcal{H}(\{0, 1\}^{\mathbb{N}})$ has topological entropy zero. Hence, by combining these two facts, we have the following result:

For the generic $h \in \mathcal{H}(\{0, 1\}^{\mathbb{N}})$, $\text{ent}(\tilde{h}) = 0$.

We remark that Theorem 1 also implies this result of Glasner and Weiss, since homeomorphisms with positive topological entropy are Li-Yorke chaotic [13]. In strong contrast, it was proved in [11] that for the generic $h \in \mathcal{H}(\{0, 1\}^{\mathbb{N}})$, $\text{ent}(\bar{h}) = \infty$.

Let us now consider the case of continuous maps. For this purpose, we will need the following lemmas.

Lemma 2. *Let \mathcal{P} be a partition of $\{0, 1\}^{\mathbb{N}}$. For every $\mu, \nu \in \mathcal{M}(\{0, 1\}^{\mathbb{N}})$, if*

$$d_P(\mu, \nu) < \delta \leq \delta(\mathcal{P}),$$

then

$$|\mu(a) - \nu(a)| < \delta \quad \text{for all } a \in \mathcal{P}.$$

Proof. Let γ be such that $d_P(\mu, \nu) < \gamma < \delta$. Since $\gamma < \delta(\mathcal{P})$, $a^\gamma = a$ for every $a \in \mathcal{P}$. Therefore, since $d_P(\mu, \nu) < \gamma$,

$$\mu(a) \leq \nu(a^\gamma) + \gamma = \nu(a) + \gamma \quad \text{and} \quad \nu(a) \leq \mu(a^\gamma) + \gamma = \mu(a) + \gamma,$$

and so $|\mu(a) - \nu(a)| \leq \gamma < \delta$ ($a \in \mathcal{P}$). \square

Lemma 3. *Let \mathcal{P} be a partition of $\{0, 1\}^\mathbb{N}$. For every $\mu, \nu \in \mathcal{M}(\{0, 1\}^\mathbb{N})$, if*

$$|\mu(a) - \nu(a)| \leq \frac{\text{mesh}(\mathcal{P})}{\text{card}(\mathcal{P})} \quad \text{for all } a \in \mathcal{P},$$

then

$$d_P(\mu, \nu) \leq \text{mesh}(\mathcal{P}).$$

Proof. Fix $\gamma > \text{mesh}(\mathcal{P})$. For each Borel subset X of $\{0, 1\}^\mathbb{N}$, $\bigcup\{a : a \in I_{\mathcal{P}}(X)\} \subset X^\gamma$, and so

$$\mu(X) = \sum_{a \in I_{\mathcal{P}}(X)} \mu(X \cap a) \leq \sum_{a \in I_{\mathcal{P}}(X)} \mu(a) \leq \left(\sum_{a \in I_{\mathcal{P}}(X)} \nu(a) \right) + \text{mesh}(\mathcal{P}) < \nu(X^\gamma) + \gamma.$$

Thus, $d_P(\mu, \nu) \leq \gamma$. Since $\gamma > \text{mesh}(\mathcal{P})$ is arbitrary, we have the desired inequality. \square

Given $f : M \rightarrow M$, recall that f is *equicontinuous* at a point $x \in M$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d(y, x) < \delta \implies d(f^n(y), f^n(x)) < \varepsilon \text{ for all } n \geq 0.$$

Theorem 4. *For the generic $f \in \mathcal{C}(\{0, 1\}^\mathbb{N})$, \tilde{f} is equicontinuous at every point.*

Proof. Let $f \in \mathcal{C}(\{0, 1\}^\mathbb{N})$ satisfy property (Q) of Theorem A. Given $\varepsilon > 0$, there exist a partition \mathcal{P} of $\{0, 1\}^\mathbb{N}$ of mesh $< \varepsilon$ and an integer $q \geq 1$ such that every component of $\text{Gr}(f, \mathcal{P})$ is a balloon of type (q, q) . Let $\mu, \nu \in \mathcal{M}(\{0, 1\}^\mathbb{N})$ be such that

$$d_P(\mu, \nu) < \min \left\{ \delta(\mathcal{P}), \frac{\text{mesh}(\mathcal{P})}{2 \text{card}(\mathcal{P})} \right\}.$$

By Lemma 2,

$$|\mu(a) - \nu(a)| < \frac{\text{mesh}(\mathcal{P})}{2 \text{card}(\mathcal{P})} \quad (a \in \mathcal{P}).$$

Fix $a \in \mathcal{P}$ and $n \geq 0$. Let B be the component (balloon) of $\text{Gr}(f, \mathcal{P})$ that contains a as a vertex. Then,

$$|(\tilde{f}^n(\mu))(a) - (\tilde{f}^n(\nu))(a)| = |\mu(f^{-n}(a)) - \nu(f^{-n}(a))| < \frac{\text{mesh}(\mathcal{P})}{\text{card}(\mathcal{P})},$$

because $f^{-n}(a)$ is empty or a vertex of B or the union of two vertices of B . Therefore, it follows from Lemma 3 that

$$d_P(\tilde{f}^n(\mu), \tilde{f}^n(\nu)) < \varepsilon \quad \text{for all } n \geq 0.$$

This completes the proof. \square

The above theorem is not true in the case of homeomorphisms. Indeed, it was proved in [10] that the generic $h \in \mathcal{H}(\{0, 1\}^\mathbb{N})$ is not equicontinuous at each point of an uncountable set, and so the same is true for the induced map \tilde{h} .

The above theorem has the following interesting consequences.

Corollary 5. *For the generic $f \in \mathcal{C}(\{0, 1\}^\mathbb{N})$, \tilde{f} has no Li-Yorke pair.*

Corollary 6. *For the generic $f \in \mathcal{C}(\{0, 1\}^\mathbb{N})$, $\text{ent}(\tilde{f}) = 0$.*

4 Chain continuity and shadowing

Given $f : M \rightarrow M$, recall that f is *chain continuous* at a point $x \in M$ [4, 7] if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any choice of points

$$x_0 \in B(x; \delta), \quad x_1 \in B(f(x_0); \delta), \quad x_2 \in B(f(x_1); \delta), \dots,$$

we have that

$$d(x_n, f^n(x)) < \varepsilon \quad \text{for all } n \geq 0.$$

Of course, chain continuity is a much stronger property than equicontinuity.

It was proved in [11] that for the generic $f \in \mathcal{C}(\{0, 1\}^{\mathbb{N}})$ (resp. $h \in \mathcal{H}(\{0, 1\}^{\mathbb{N}})$), the induced map \bar{f} (resp. \bar{h}) is chain continuous at every point (resp. is chain continuous at every point of a dense open set). We shall see that the situation is completely different for the induced map \tilde{f} (resp. \tilde{h}). Indeed, for the generic $f \in \mathcal{C}(\{0, 1\}^{\mathbb{N}})$ (resp. $h \in \mathcal{H}(\{0, 1\}^{\mathbb{N}})$), the induced map \tilde{f} (resp. \tilde{h}) has no point of chain continuity. We shall obtain these facts from more general results for arbitrary compact metric spaces.

Lemma 7. *If $\mu, \nu \in \mathcal{M}(M)$, $\delta > 0$, $n \in \mathbb{N}_0$ and $1 - (n + 1)\delta > 0$, then*

$$d_P((1 - (n + 1)\delta)\mu + (n + 1)\delta\nu, (1 - n\delta)\mu + n\delta\nu) \leq \delta.$$

Proof. For each Borel subset X of M ,

$$\begin{aligned} ((1 - (n + 1)\delta)\mu + (n + 1)\delta\nu)(X) &= ((1 - n\delta)\mu + n\delta\nu)(X) + \delta(\nu(X) - \mu(X)) \\ &\leq ((1 - n\delta)\mu + n\delta\nu)(X^\delta) + \delta. \end{aligned}$$

This implies the desired inequality. □

Theorem 8. *Given $0 < \delta < 1$, there exists $k_0 \in \mathbb{N}$ such that for any $f \in \mathcal{C}(M)$, any $\mu, \nu \in \mathcal{M}(M)$ and any $k \geq k_0$, there exist*

$$\mu_1 \in B(\tilde{f}(\mu); \delta), \quad \mu_2 \in B(\tilde{f}(\mu_1); \delta), \dots, \mu_k \in B(\tilde{f}(\mu_{k-1}); \delta)$$

such that

$$\mu_k = \tilde{f}^k(\nu).$$

In particular, if we choose $\nu = \pi_z$ for some $z \in M$, then

$$\mu_k = \pi_{f^k(z)}.$$

Proof. Let $0 < \gamma < \delta$ and let $k_0 \in \mathbb{N}$ be such that $(k_0 - 1)\gamma < 1 \leq k_0\gamma$. Define

$$\mu_1 := (1 - \gamma)\tilde{f}(\mu) + \gamma\tilde{f}(\nu).$$

By Lemma 7, $d_P(\mu_1, \tilde{f}(\mu)) \leq \gamma < \delta$. Moreover,

$$\tilde{f}(\mu_1) = (1 - \gamma)\tilde{f}^2(\mu) + \gamma\tilde{f}^2(\nu).$$

Define

$$\mu_2 := (1 - 2\gamma)\tilde{f}^2(\mu) + 2\gamma\tilde{f}^2(\nu).$$

By Lemma 7, $d_P(\mu_2, \tilde{f}(\mu_1)) \leq \gamma < \delta$. Moreover,

$$\tilde{f}(\mu_2) = (1 - 2\gamma)\tilde{f}^3(\mu) + 2\gamma\tilde{f}^3(\nu).$$

We continue this process until we define

$$\mu_{k_0-1} := (1 - (k_0 - 1)\gamma)\tilde{f}^{k_0-1}(\mu) + (k_0 - 1)\gamma\tilde{f}^{k_0-1}(\nu).$$

Then,

$$\tilde{f}(\mu_{k_0-1}) = (1 - (k_0 - 1)\gamma)\tilde{f}^{k_0}(\mu) + (k_0 - 1)\gamma\tilde{f}^{k_0}(\nu).$$

Now, we define

$$\mu_{k_0} := \tilde{f}^{k_0}(\nu).$$

For each Borel subset X of M ,

$$(\tilde{f}(\mu_{k_0-1}))(X) \leq 1 - (k_0 - 1)\gamma + (k_0 - 1)\gamma(\tilde{f}^{k_0}(\nu))(X) \leq \mu_{k_0}(X^\gamma) + \gamma.$$

Thus, $d_P(\mu_{k_0}, \tilde{f}(\mu_{k_0-1})) \leq \gamma < \delta$. Finally, it is enough to complete the sequence by defining $\mu_{k_0+1} := \tilde{f}(\mu_{k_0}), \dots, \mu_k := \tilde{f}(\mu_{k-1})$. \square

Theorem 9. *Given $0 < \delta < 1$, there exists $k_0 \in \mathbb{N}$ such that for any $h \in \mathcal{H}(M)$, any $\mu, \nu \in \mathcal{M}(M)$ and any $k \geq k_0$, there exist*

$$\mu_1 \in B(\tilde{h}(\mu); \delta), \mu_2 \in B(\tilde{h}(\mu_1); \delta), \dots, \mu_k \in B(\tilde{h}(\mu_{k-1}); \delta)$$

such that

$$\mu_k = \nu.$$

Proof. Let $k_0 \in \mathbb{N}$ be as in Theorem 8. Since $h \in \mathcal{H}(M)$, we can choose $\nu' \in \mathcal{M}(M)$ such that $\tilde{h}^k(\nu') = \nu$. Hence, it is enough to consider ν' in place of ν in Theorem 8. \square

The next result characterizes the chain continuity of the induced map \tilde{f} .

Theorem 10. *For every $f \in \mathcal{C}(M)$, the following assertions are equivalent:*

- (i) \tilde{f} is chain continuous at some point;
- (ii) \tilde{f} is chain continuous at every point;
- (iii) $\bigcap_{n=1}^{\infty} f^n(M)$ is a singleton.

Proof. (i) \Rightarrow (iii): Put $Y := \bigcap_{n=1}^{\infty} f^n(M)$. By hypothesis, there exists $\mu \in \mathcal{M}(M)$ such that \tilde{f} is chain continuous at μ . Fix $0 < \varepsilon < 1/2$. There exists $0 < \delta < 1$ such that the relations

$$\mu_0 \in B(\mu; \delta), \mu_1 \in B(\tilde{f}(\mu_0); \delta), \mu_2 \in B(\tilde{f}(\mu_1); \delta), \dots$$

imply

$$d_P(\mu_n, \tilde{f}^n(\mu)) < \varepsilon \quad \text{for all } n \geq 0.$$

Let $k_0 \in \mathbb{N}$ be associated to this δ as in Theorem 8. Given $y \in Y$, we can choose $z \in M$ such that $f^{k_0}(z) = y$. By Theorem 8 with $\nu = \pi_z$ and $k = k_0$, there exist

$$\mu_1 \in B(\tilde{f}(\mu); \delta), \mu_2 \in B(\tilde{f}(\mu_1); \delta), \dots, \mu_{k_0} \in B(\tilde{f}(\mu_{k_0-1}); \delta)$$

with

$$\mu_{k_0} = \pi_{f^{k_0}(z)} = \pi_y.$$

Hence, $d_P(\pi_y, \tilde{f}^{k_0}(\mu)) < \varepsilon$. Since $y \in Y$ is arbitrary, we conclude that

$$d_P(\pi_y, \pi_w) < 2\varepsilon \quad \text{whenever } y, w \in Y.$$

This implies that $\text{diam}(Y) < 2\varepsilon$, which proves (iii).

(iii) \Rightarrow (ii): Let $\bigcap_{n=1}^{\infty} f^n(M) = \{a\}$. We claim that

$$\bigcap_{n=1}^{\infty} \tilde{f}^n(\mathcal{M}(M)) = \{\pi_a\}.$$

Indeed, since $\tilde{f}(\pi_a) = \pi_{f(a)} = \pi_a$, it is clear that π_a belongs to the above intersection. Conversely, let ν be an element of the above intersection. Then, for each $n \in \mathbb{N}$, there exists $\mu_n \in \mathcal{M}(M)$ such that $\nu = \tilde{f}^n(\mu_n)$. Hence,

$$\nu(f^n(M)) = \mu_n(f^{-n}(f^n(M))) = \mu_n(M) = 1 \quad \text{for all } n \in \mathbb{N},$$

which implies that $\nu(\{a\}) = 1$, that is, $\nu = \pi_a$. Now, (ii) follows from Lemma 11 below.

(ii) \Rightarrow (i): Obvious. \square

Lemma 11. *If $f \in \mathcal{C}(M)$ and $\bigcap_{n=1}^{\infty} f^n(M)$ is a singleton, then f is chain continuous at every point.*

Proof. Assume $\bigcap_{n=1}^{\infty} f^n(M) = \{a\}$. Then a is a fixed point of f that uniformly attracts all orbits. Hence, it is clear that f is equicontinuous at every point. Fix $\varepsilon > 0$ and let $\gamma > 0$ be such that the relation $d(y, a) < \gamma$ implies $d(f^n(y), a) < \varepsilon/3$ for all $n \geq 0$. Let $k \in \mathbb{N}$ be such that $f^k(M) \subset B(a; \gamma)$. Then,

$$d(f^n(y), a) < \frac{\varepsilon}{3} \quad \text{for all } y \in M \text{ and all } n \geq k. \quad (6)$$

Moreover, there exists $\delta > 0$ such that the following property holds:

(*) For every $y \in M$ and for every choice of points

$$y_0 \in B(y; \delta), y_1 \in B(f(y_0); \delta), \dots, y_k \in B(f(y_{k-1}); \delta),$$

we have that $y_k \in B(a; \gamma)$ and $d(y_j, f^j(y)) < \varepsilon/3$ for all $j \in \{0, 1, \dots, k\}$.

Take $x \in M$ and let $x_0 \in B(x; \delta)$, $x_1 \in B(f(x_0); \delta)$, $x_2 \in B(f(x_1); \delta), \dots$. We have to prove that

$$d(x_n, f^n(x)) < \varepsilon \quad \text{for all } n \geq 0. \quad (7)$$

By (*), $x_k \in B(a; \gamma)$ and the inequality in (7) holds for every $n \in \{0, 1, \dots, k\}$. Since $d(x_k, a) < \gamma$,

$$d(f^n(x_k), a) < \frac{\varepsilon}{3} \quad \text{for all } n \geq 0. \quad (8)$$

Moreover, by applying (*) with $y = y_0 = x_k, y_1 = x_{k+1}, \dots, y_k = x_{2k}$, we see that $x_{2k} \in B(a; \gamma)$ and

$$d(x_n, f^{n-k}(x_k)) < \frac{\varepsilon}{3} \quad \text{for all } n \in \{k+1, \dots, 2k\}. \quad (9)$$

By (6), (8) and (9), the inequality in (7) holds for every $n \in \{k+1, \dots, 2k\}$. Since $d(x_{2k}, a) < \gamma$, we can repeat the argument and conclude that $x_{3k} \in B(a; \gamma)$ and the inequality in (7) holds for every $n \in \{2k+1, \dots, 3k\}$. By continuing this process, we obtain the desired result. \square

For the generic $f \in \mathcal{C}(\{0,1\}^{\mathbb{N}})$, since f has no periodic point, it follows from Theorem 10 that \tilde{f} has no point of chain continuity.

Theorem 12. *Suppose that M has at least two points. For every $h \in \mathcal{H}(M)$, \tilde{h} has no point of chain continuity.*

Proof. Follows immediately from Theorem 10. \square

Given $f \in \mathcal{C}(M)$, recall that f is *topologically transitive* (resp. *mixing*) if, for any pair $U, V \subset M$ of nonempty open sets, there exists $k \in \mathbb{N}_0$ (resp. $k_0 \in \mathbb{N}_0$) such that $f^k(U) \cap V \neq \emptyset$ (resp. for all $k \geq k_0$).

Given $f \in \mathcal{C}(M)$ and $\delta > 0$, recall that a finite sequence $(x_n)_{n=0,1,\dots,k}$ of elements of M is a δ -chain from x_0 to x_k if $d(f(x_n), x_{n+1}) < \delta$ for all $n = 0, 1, \dots, k-1$. In this case, we say that k is the *length* of the chain. Recall that f is *chain mixing* if for every $\delta > 0$ and for every pair $x, y \in M$, there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, there exists a δ -chain from x to y of length k . Note that if f is chain mixing, then f is necessarily surjective.

Theorem 13. *For every $h \in \mathcal{H}(M)$, \tilde{h} is chain mixing.*

Proof. Follows immediately from Theorem 9. \square

In view of the above theorem, it is natural to ask if \tilde{h} is always mixing. Let us see that this is not the case. Indeed, it was proved in [6] that \tilde{h} topologically transitive implies h topologically transitive. As a consequence, for the generic $h \in \mathcal{H}(\{0,1\}^{\mathbb{N}})$, \tilde{h} is not topologically transitive.

On the other hand, for the generic $f \in \mathcal{C}(\{0,1\}^{\mathbb{N}})$, since f is not surjective, it follows that \tilde{f} is neither topologically transitive nor chain mixing.

Given $h \in \mathcal{H}(M)$, recall that a sequence $(x_n)_{n \in \mathbb{Z}}$ is a δ -pseudotrajectory ($\delta > 0$) of h if

$$d(h(x_n), x_{n+1}) \leq \delta \quad \text{for all } n \in \mathbb{Z}.$$

The homeomorphism h has the *weak shadowing property* [17] if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every δ -pseudotrajectory $(x_n)_{n \in \mathbb{Z}}$ of h there exists $x \in M$ such that

$$\{x_n : n \in \mathbb{Z}\} \subset \{h^n(x) : n \in \mathbb{Z}\}^\varepsilon.$$

Moreover, h has the *shadowing property* [15, 16] if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every δ -pseudotrajectory $(x_n)_{n \in \mathbb{Z}}$ of h is ε -shadowed by a real trajectory of h , i.e., there exists $x \in M$ such that

$$d(x_n, h^n(x)) < \varepsilon \quad \text{for all } n \in \mathbb{Z}.$$

It was proved in [11] that for the generic $h \in \mathcal{H}(\{0,1\}^{\mathbb{N}})$, the induced map \bar{h} has the shadowing property. Again, we shall see that the induced map \tilde{h} has a completely different behaviour.

Theorem 14. *For the generic $h \in \mathcal{H}(\{0,1\}^{\mathbb{N}})$, \tilde{h} does not have the weak shadowing property.*

Proof. Fix a generic $h \in \mathcal{H}(\{0, 1\}^{\mathbb{N}})$ and suppose that \tilde{h} has the weak shadowing property. Let U, V be a pair of nonempty open sets in $\mathcal{M}(\{0, 1\}^{\mathbb{N}})$. Fix $\mu \in U$ and $\nu \in V$, and choose $\varepsilon > 0$ such that

$$B(\mu; \varepsilon) \subset U \quad \text{and} \quad B(\nu; \varepsilon) \subset V.$$

Since \tilde{h} has the weak shadowing property, there is a $\delta > 0$ associated to this ε according to the definition of weak shadowing. Since \tilde{h} is chain mixing (Theorem 13), there is a δ -chain $(\mu_0, \mu_1, \dots, \mu_k)$ of \tilde{h} starting at $\mu_0 = \mu$ and ending at $\mu_k = \nu$. Of course, we can extend this δ -chain to a δ -pseudotrajectory $(\mu_n)_{n \in \mathbb{Z}}$ of \tilde{h} . By weak shadowing, there exists $\eta \in \mathcal{M}(\{0, 1\}^{\mathbb{N}})$ such that

$$\{\mu_n : n \in \mathbb{Z}\} \subset \{\tilde{h}^n(\eta) : n \in \mathbb{Z}\}^\varepsilon.$$

In particular, there are $n_1, n_2 \in \mathbb{Z}$ such that $\tilde{h}^{n_1}(\eta) \in U$ and $\tilde{h}^{n_2}(\eta) \in V$, and so $\tilde{h}^{n_2-n_1}(U) \cap V \neq \emptyset$. This implies that \tilde{h} is topologically transitive. As observed after the proof of Theorem 13, this is a contradiction. \square

Remark 15. The above proof actually establishes the following more general result:

If $h \in \mathcal{H}(M)$ is not topologically transitive, then \tilde{h} does not have the weak shadowing property.

5 Recurrence

Given a map $f : M \rightarrow M$, we denote by $P(f)$ (resp. $R(f)$, $\Omega(f)$, $CR(f)$), the set of all periodic (resp. recurrent, nonwandering, chain recurrent) points of f .

Given partitions \mathcal{P} and \mathcal{Q} of $\{0, 1\}^{\mathbb{N}}$, we say that \mathcal{Q} *strongly refines* \mathcal{P} if each $a \in \mathcal{Q}$ is properly contained in some $a' \in \mathcal{P}$. A sequence (\mathcal{P}_n) of partitions of $\{0, 1\}^{\mathbb{N}}$ is said to be *strongly decreasing* if \mathcal{P}_{n+1} strongly refines \mathcal{P}_n for all n . Recall that (\mathcal{P}_n) is said to be *null* if $\text{mesh}(\mathcal{P}_n) \rightarrow 0$ as $n \rightarrow \infty$ [10]. Note that every null sequence of partitions of $\{0, 1\}^{\mathbb{N}}$ has a strongly decreasing (and null) subsequence.

Theorem 16. *For the generic $f \in \mathcal{C}(\{0, 1\}^{\mathbb{N}})$, the following properties hold:*

- (a) \tilde{f} has uncountably many periodic points of each period $p \geq 1$.
- (b) Any neighborhood of any periodic point of \tilde{f} of period p contains uncountably many periodic points of \tilde{f} of period kp , for each $k \in \mathbb{N}$.
- (c) $R(\tilde{f}) = \Omega(\tilde{f}) = CR(\tilde{f})$.
- (d) $CR(\tilde{f})$ has empty interior in $\tilde{f}(\mathcal{M}(\{0, 1\}^{\mathbb{N}}))$.
- (e) $P(\tilde{f})$ is dense in $CR(\tilde{f})$.

Proof. Let $f \in \mathcal{C}(\{0, 1\}^{\mathbb{N}})$ satisfy property (Q). We shall divide the proof in seven steps.

Step 1. Let \mathcal{P} be a partition of $\{0, 1\}^{\mathbb{N}}$ such that every component of $\text{Gr}(f, \mathcal{P})$ is a balloon of a certain type $(q!, q!)$. Choose one such component B ; say

$$B = \{v_1, \dots, v_{q!}\} \cup \{w_1, \dots, w_{q!}\},$$

with usual labeling. Choose also a vertex $w \in \{w_1, \dots, w_{q!}\}$ and an integer $p \in \mathbb{N}$. Let $k \in \mathbb{N}$ be the smallest integer such that

$$f^{kp}(w) \subset w.$$

Let $a_0, \dots, a_{k-1} \in \{w_1, \dots, w_{q!}\}$ be determined by the relations

$$f^{jp}(w) \subset a_j, \quad j = 0, \dots, k-1.$$

Then, there are uncountably many periodic points μ of \tilde{f} of period p such that

$$\mu(a_0) = \mu(a_1) = \dots = \mu(a_{k-1}) \quad (10)$$

and

$$\mu(a) = 0 \quad \text{for all } a \in \mathcal{P} \setminus \{a_0, a_1, \dots, a_{k-1}\}. \quad (11)$$

Indeed, let us fix a strongly decreasing null sequence $(\mathcal{P}_n)_{n \in \mathbb{N}}$ of partitions of $\{0, 1\}^{\mathbb{N}}$ such that each component of $\text{Gr}(f, \mathcal{P}_n)$ is a balloon of type $(q_n!, q_n!)$, \mathcal{P}_1 refines \mathcal{P} and $q_1 > p$. By an *admissible sequence* we mean a sequence $\mathcal{B} := (B_n)_{n \in \mathbb{N}}$, where each B_n is a component (balloon) of $\text{Gr}(f, \mathcal{P}_n)$, such that the initial vertex of B_1 is contained in the initial vertex of B and the initial vertex of B_{n+1} is contained in the initial vertex of B_n for each $n \in \mathbb{N}$. To each admissible sequence \mathcal{B} , we shall associate a measure $\mu_{\mathcal{B}} \in \mathcal{M}(\{0, 1\}^{\mathbb{N}})$ which will be constructed as follows. Write

$$B_n = \{v_{n,1}, \dots, v_{n,q_n!}\} \cup \{w_{n,1}, \dots, w_{n,q_n!}\},$$

with usual labeling. We extend the “loop” $\{w_{n,1}, \dots, w_{n,q_n!}\}$ to a sequence $(w_{n,j})_{j \in \mathbb{N}}$ by considering $w_{n,i} = w_{n,j}$ whenever $i \equiv j \pmod{q_n!}$. It is easy to verify that the collection

$$\mathcal{S} := \{\emptyset\} \cup \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \cup \dots$$

is a semiring of subsets of $\{0, 1\}^{\mathbb{N}}$ (i.e., $a, b \in \mathcal{S}$ implies that $a \cap b \in \mathcal{S}$ and that $a \setminus b$ is a finite union of pairwise disjoint elements of \mathcal{S}). Since \mathcal{P}_1 refines \mathcal{P} and $v_{1,1} \subset v_1$, it follows that

$$q_1 \geq q \quad \text{and} \quad w_{1,1} \subset w_1.$$

Moreover, since \mathcal{P}_{n+1} refines \mathcal{P}_n and $v_{n+1,1} \subset v_{n,1}$, we also have that

$$q_{n+1} \geq q_n \quad \text{and} \quad w_{n+1,1} \subset w_{n,1} \quad (n \in \mathbb{N}).$$

As a consequence, there is a smallest $t \in \mathbb{N}$ such that

$$w_{n,t} \subset w \quad \text{for all } n \in \mathbb{N}.$$

We define a set function $\varphi : \mathcal{S} \rightarrow [0, 1]$ by

$$\varphi(a) := \frac{p}{q_n!} \quad \text{if } a = w_{n,t+jp} \text{ for some } n \in \mathbb{N} \text{ and some } 0 \leq j \leq \frac{q_n!}{p} - 1$$

and

$$\varphi(a) := 0 \quad \text{otherwise.}$$

We claim that

$$\varphi(a) = \sum_{b \in I_{\mathcal{P}_{n+1}}(a)} \varphi(b), \quad (12)$$

for every $n \in \mathbb{N}$ and every $a \in \mathcal{P}_n$. Indeed, for each $0 \leq j \leq \frac{q_n!}{p} - 1$,

$$w_{n+1,t+ip} \subset w_{n,t+jp} \iff i = j + \ell \frac{q_n!}{p} \text{ for some } 0 \leq \ell \leq \frac{q_{n+1}!}{q_n!} - 1.$$

Therefore,

$$\sum_{b \in I_{\mathcal{P}_{n+1}}(w_{n,t+jp})} \varphi(b) = \sum_{\ell=0}^{\frac{q_{n+1}!}{q_n!}-1} \varphi(w_{n+1,t+jp+\ell q_n!}) = \frac{q_{n+1}!}{q_n!} \cdot \frac{p}{q_{n+1}!} = \frac{p}{q_n!} = \varphi(w_{n,t+jp}).$$

On the other hand, if $a \neq w_{n,t+jp}$ for all $0 \leq j \leq \frac{q_n!}{p} - 1$, then no set of the form $w_{n+1,t+ip}$ ($0 \leq i \leq \frac{q_{n+1}!}{p} - 1$) is contained in a , and so

$$\varphi(a) = 0 = \sum_{b \in I_{\mathcal{P}_{n+1}}(a)} \varphi(b).$$

This completes the proof of our claim.

Let us now prove that φ is finitely additive. Let $a \in \mathcal{S}$ be nonempty and assume that a is the union of a finite collection \mathcal{C} of pairwise disjoint nonempty elements of \mathcal{S} . We have to prove that

$$\varphi(a) = \sum_{c \in \mathcal{C}} \varphi(c). \quad (13)$$

Since this is obvious if $\mathcal{C} = \{a\}$, let us assume that this is not the case. Let $n \in \mathbb{N}$ be such that $a \in \mathcal{P}_n$ and let $m \in \mathbb{N}$ be the largest positive integer such that $\mathcal{C} \cap \mathcal{P}_{n+m} \neq \emptyset$. Define

$$\mathcal{C}_j := \mathcal{C} \cap \mathcal{P}_{n+j} \quad \text{for } j = 1, \dots, m.$$

Then $\mathcal{C} = \mathcal{C}_1 \cup \dots \cup \mathcal{C}_m$. We shall prove (13) by induction on m . If $m = 1$ then $\mathcal{C} = \mathcal{C}_1 = I_{\mathcal{P}_{n+1}}(a)$, and so (13) follows from (12). Assume $m \geq 2$ and the result true with $m - 1$ in place of m . Choose $b \in \mathcal{C}_m$. There is a unique $b' \in \mathcal{P}_{n+m-1}$ such that $b \subset b'$. Since \mathcal{P}_{n+m} strongly refines \mathcal{P}_{n+m-1} , $b' \neq b$. Moreover, since $b \subset a$, we must have $b' \subset a$. Thus, $b' \notin \mathcal{C}$ and $I_{\mathcal{P}_{n+m}}(b') \subset \mathcal{C}$. We define

$$\mathcal{C}' := (\mathcal{C} \setminus I_{\mathcal{P}_{n+m}}(b')) \cup \{b'\}.$$

Then, $a = \cup \mathcal{C}'$ (with disjoint union) and $\sum_{c' \in \mathcal{C}'} \varphi(c') = \sum_{c \in \mathcal{C}} \varphi(c)$ because of (12). We can repeat this argument until we obtain a finite collection \mathcal{D} of pairwise disjoint nonempty elements of \mathcal{S} such that

$$a = \cup \mathcal{D}, \quad \sum_{d \in \mathcal{D}} \varphi(d) = \sum_{c \in \mathcal{C}} \varphi(c) \quad \text{and} \quad \mathcal{D} \cap \mathcal{P}_{n+j} = \emptyset \quad \text{for all } j \geq m.$$

By the induction hypothesis,

$$\sum_{c \in \mathcal{C}} \varphi(c) = \sum_{d \in \mathcal{D}} \varphi(d) = \varphi(a),$$

as was to be shown.

Since the elements of \mathcal{S} are clopen, it is not possible to write an element of \mathcal{S} as a countably infinite union of pairwise disjoint nonempty elements of \mathcal{S} . As a consequence, φ

is countably additive. By the extension theorem of measure theory, there exists a measure $\mu_{\mathcal{B}}$ defined on a σ -algebra \mathcal{A} containing \mathcal{S} that extends φ . Since every open subset of $\{0, 1\}^{\mathbb{N}}$ can be written as a countable union of elements of \mathcal{S} , \mathcal{A} contains the Borel subsets of $\{0, 1\}^{\mathbb{N}}$. Hence, we may regard $\mu_{\mathcal{B}}$ as a Borel measure. Since

$$\mu_{\mathcal{B}}(\{0, 1\}^{\mathbb{N}}) = \sum_{a \in \mathcal{P}_1} \mu_{\mathcal{B}}(a) = \sum_{a \in \mathcal{P}_1} \varphi(a) = \sum_{j=0}^{\frac{q_1!}{p}-1} \varphi(w_{1,t+jp}) = \sum_{j=0}^{\frac{q_1!}{p}-1} \frac{p}{q_1!} = 1,$$

we see that $\mu_{\mathcal{B}}$ is a probability measure. In other words,

$$\mu_{\mathcal{B}} \in \mathcal{M}(\{0, 1\}^{\mathbb{N}}).$$

For each $n \in \mathbb{N}$, since $\mu_{\mathcal{B}}(f^{-p}(a)) = \mu_{\mathcal{B}}(a)$ for all $a \in \mathcal{P}_n$, it follows from Lemma 3 that

$$d_P(\tilde{f}^p(\mu_{\mathcal{B}}), \mu_{\mathcal{B}}) \leq \text{mesh } \mathcal{P}_n.$$

Since $\text{mesh } \mathcal{P}_n \rightarrow 0$ as $n \rightarrow \infty$, we obtain

$$\tilde{f}^p(\mu_{\mathcal{B}}) = \mu_{\mathcal{B}}.$$

On the other hand, $\mu_{\mathcal{B}}(f^{-j}(w_{1,t+ip})) = 0 \neq p/q_1! = \mu_{\mathcal{B}}(w_{1,t+ip})$ for each $1 \leq j < p$, which implies that

$$\tilde{f}^j(\mu_{\mathcal{B}}) \neq \mu_{\mathcal{B}} \quad \text{for each } 1 \leq j < p.$$

Therefore, $\mu_{\mathcal{B}}$ is a periodic point of \tilde{f} of period p .

If $\mathcal{B}' := (B'_n)_{n \in \mathbb{N}}$ is an admissible sequence with $\mathcal{B}' \neq \mathcal{B}$, then $B'_m \neq B_m$ for some $m \in \mathbb{N}$, and so $\mu_{\mathcal{B}}(\cup B_m) = 1$ whereas $\mu_{\mathcal{B}'}(\cup B_m) = 0$. This shows that distinct admissible sequences generate distinct probability measures. Since the set of all admissible sequences is uncountable (by a simple diagonal argument), we conclude that

$$\{\mu_{\mathcal{B}} : \mathcal{B} \text{ is an admissible sequence}\}$$

is an uncountable set of periodic points of \tilde{f} of period p . By construction, it is easy to see that each $\mu_{\mathcal{B}}$ satisfies (10) and (11). Thus, the proof of Step 1 is complete.

Step 2. Proof of (a).

Property (a) follows immediately from Step 1.

Step 3. Let \mathcal{P} be a partition of $\{0, 1\}^{\mathbb{N}}$ such that every component of $\text{Gr}(f, \mathcal{P})$ is a balloon of type $(q!, q!)$. Let

$$B_i := \{v_{i,1}, \dots, v_{i,q!}\} \cup \{w_{i,1}, \dots, w_{i,q!}\} \quad (1 \leq i \leq N)$$

be the components (balloons) of $\text{Gr}(f, \mathcal{P})$. For every $\mu \in CR(\tilde{f})$,

$$\mu(v_{i,j}) = 0 \quad \text{for all } i \text{ and } j.$$

Choose $0 < \delta < \delta(\mathcal{P})$. Since μ is a chain recurrent point of \tilde{f} , there is a δ -chain $(\mu_n)_{n=0,1,\dots,k}$ from $\mu_0 := \mu$ to $\mu_k := \mu$. Since $d_P(\tilde{f}(\mu_n), \mu_{n+1}) < \delta$, Lemma 2 gives

$$|\mu_n(f^{-1}(a)) - \mu_{n+1}(a)| < \delta \quad \text{for all } a \in \mathcal{P} \quad (0 \leq n \leq k-1).$$

Since $f^{-q!}(v_{i,j}) = \emptyset$, it follows that

$$\mu(v_{i,j}) < q!\delta \quad (1 \leq i \leq N, 1 \leq j \leq q!).$$

Since $\delta > 0$ can be chosen arbitrarily small, $\mu(v_{i,j}) = 0$ for each i and j .

Step 4. Proof of (c).

Fix $\mu \in CR(\tilde{f})$. Given $\varepsilon > 0$, let \mathcal{P} be a partition of $\{0, 1\}^{\mathbb{N}}$ of mesh $< \varepsilon$ such that every component of $\text{Gr}(f, \mathcal{P})$ is a balloon of type $(q!, q!)$. Let

$$B_i := \{v_{i,1}, \dots, v_{i,q!}\} \cup \{w_{i,1}, \dots, w_{i,q!}\} \quad (1 \leq i \leq N)$$

be the components (balloons) of $\text{Gr}(f, \mathcal{P})$. Since $f^{-q!}(v_{i,j}) = \emptyset$ and $f^{-q!}(w_{i,j}) = v_{i,j} \cup w_{i,j}$, it follows from Step 3 that

$$\mu(f^{-q!}(a)) = \mu(a) \quad \text{for all } a \in \mathcal{P}.$$

Thus, by Lemma 3, $d_P(\tilde{f}^{q!}(\mu), \mu) < \varepsilon$. This proves that $\mu \in R(\tilde{f})$.

Step 5. Proof of (d).

Let $\mu := \tilde{f}(\nu)$ be an arbitrary element in the range of \tilde{f} . Let \mathcal{P} be a partition of $\{0, 1\}^{\mathbb{N}}$ such that every component of $\text{Gr}(f, \mathcal{P})$ is a balloon of type $(q!, q!)$ with $q \geq 2$. Choose one such component

$$B := \{v_1, \dots, v_{q!}\} \cup \{w_1, \dots, w_{q!}\}$$

and choose a point $z \in v_1$. For each $\lambda \in (0, 1)$, define

$$\mu_\lambda := (1 - \lambda)\mu + \lambda\pi_{f(z)}.$$

Note that each μ_λ belongs to the range of \tilde{f} because $\mu_\lambda = \tilde{f}((1 - \lambda)\nu + \lambda\pi_z)$. Since $\mu_\lambda(v_2) \geq \lambda > 0$, it follows from Step 3 that $\mu_\lambda \notin CR(\tilde{f})$. Moreover, by Lemma 7, $d_P(\mu_\lambda, \mu) \leq \lambda$. Thus, there are points of $\tilde{f}(\mathcal{M}(\{0, 1\}^{\mathbb{N}})) \setminus CR(\tilde{f})$ arbitrarily close to μ .

Step 6. Proof of (e).

Fix $\mu \in R(\tilde{f})$ and $\varepsilon > 0$. Let \mathcal{P} and B_1, \dots, B_N be as in Step 4. Define

$$\delta := \min \left\{ \delta(\mathcal{P}), \frac{\text{mesh}(\mathcal{P})}{2q! \text{card}(\mathcal{P})} \right\}.$$

Since μ is a recurrent point of \tilde{f} , there exists $p \in \mathbb{N}$ such that $d_P(\tilde{f}^p(\mu), \mu) < \delta$. By Lemma 2,

$$|\mu(f^{-p}(a)) - \mu(a)| < \delta \quad \text{for all } a \in \mathcal{P}. \quad (14)$$

Moreover, by Step 3,

$$\mu(v_{i,j}) = 0 \quad \text{for all } i \text{ and } j. \quad (15)$$

Define

$$W_i := w_{i,1} \cup \dots \cup w_{i,q!} \quad (1 \leq i \leq N).$$

If a is some $w_{i,j}$, then the sequence

$$a, f^{-p}(a) \cap W_i, f^{-2p}(a) \cap W_i, \dots$$

is periodic. Let k be the period of this sequence. Note that k does not depend on i or j . Hence, each “loop” $\{w_{i,1}, \dots, w_{i,q!}\}$ can be partitioned in sets

$$\{a_{i,r,0}, a_{i,r,1}, \dots, a_{i,r,k-1}\} \quad (1 \leq r \leq q!/k)$$

satisfying

$$a_{i,r,t} = f^{-tp}(a_{i,r,0}) \cap W_i \quad \text{for } 1 \leq t \leq k-1$$

and

$$a_{i,r,0} = f^{-kp}(a_{i,r,0}) \cap W_i.$$

By (14) and (15),

$$|\mu(a_{i,r,t}) - \mu(a_{i,r,0})| < t\delta < q!\delta \quad \text{for all } 1 \leq t \leq k-1. \quad (16)$$

Define

$$d_{i,r} := \mu(a_{i,r,0}) + \mu(a_{i,r,1}) + \dots + \mu(a_{i,r,k-1}).$$

It follows from (16) that

$$\left| \frac{d_{i,r}}{k} - \mu(a_{i,r,0}) \right| < q!\delta. \quad (17)$$

By Step 1, for each $1 \leq i \leq N$ and each $1 \leq r \leq q!/k$, there are uncountably many periodic points $\mu_{i,r}$ of \tilde{f} of period p such that

$$\mu_{i,r}(a_{i,r,0}) = \mu_{i,r}(a_{i,r,1}) = \dots = \mu_{i,r}(a_{i,r,k-1})$$

and

$$\mu_{i,r}(a) = 0 \quad \text{for all } a \in \mathcal{P} \setminus \{a_{i,r,0}, a_{i,r,1}, \dots, a_{i,r,k-1}\}.$$

Without loss of generality, we may assume $d_{1,1} \neq 0$. We fix one such periodic point $\mu_{i,r}$ for each $(i,r) \neq (1,1)$ and consider the uncountably many possibilities for the periodic point $\mu_{1,1}$. In this way we have uncountably many periodic points of \tilde{f} of the form

$$\mu' := \sum_{i=1}^N \sum_{r=1}^{q!/k} d_{i,r} \mu_{i,r}, \quad (18)$$

satisfying

$$\tilde{f}^p(\mu') = \mu'.$$

Given $1 \leq j < p$, it is not possible that two of these periodic points have period j . Indeed, assume that

$$d_{1,1}\mu_{1,1} + \sum_{(i,r) \neq (1,1)} d_{i,r}\mu_{i,r} \quad \text{and} \quad d_{1,1}\mu'_{1,1} + \sum_{(i,r) \neq (1,1)} d_{i,r}\mu_{i,r}$$

have period j . Then

$$\tilde{f}^j(\mu_{1,1}) - \tilde{f}^j(\mu'_{1,1}) = \mu_{1,1} - \mu'_{1,1}. \quad (19)$$

By the way the measures are constructed in Step 1, $\mu_{1,1}$ and $\mu'_{1,1}$ correspond to distinct admissible sequences and so there is a Borel set b in $\{0,1\}^{\mathbb{N}}$ such that

$$\mu_{1,1}(b) > 0, \quad \mu_{1,1}(f^{-j}(b)) = 0 \quad \text{and} \quad \mu'_{1,1}(b) = \mu'_{1,1}(f^{-j}(b)) = 0,$$

which contradicts (19). Thus, uncountably many measures of the form (18) are periodic points of \tilde{f} of period p . Let μ' be as in (18). We shall prove that

$$|\mu'(a) - \mu(a)| < \frac{\text{mesh}(\mathcal{P})}{\text{card}(\mathcal{P})} \quad \text{for all } a \in \mathcal{P}. \quad (20)$$

Indeed, if a is some $v_{i,j}$, then $\mu'(a) = 0 = \mu(a)$ because of (15). If a is some $w_{i,j}$, then there are unique $1 \leq r \leq q!/k$ and $0 \leq t \leq k-1$ such that $a = a_{i,r,t}$. Hence,

$$\mu'(a) = d_{i,r}\mu_{i,r}(a) = \frac{d_{i,r}}{k}.$$

By (16) and (17), $|\mu'(a) - \mu(a)| < 2q!\delta$. By our choice of δ , we also obtain the inequality in (20) in this case. Therefore, Lemma 3 tell us that $d_P(\mu', \mu) < \varepsilon$, as was to be shown.

Step 7. Proof of (b).

In the proof of Step 6, if μ is already a periodic point of \tilde{f} , of period t say, then the p can be chosen to be any multiple kt of t , and so we obtain uncountably many periodic points of \tilde{f} of period kt in the ε -neighborhood of μ . \square

Let us now see that a result similar to Theorem 16 holds in the case of homeomorphisms.

Theorem 17. *For the generic $h \in \mathcal{H}(\{0,1\}^{\mathbb{N}})$, the following properties hold:*

- (a) \tilde{h} has uncountably many periodic points of each period $p \geq 1$.
- (b) Any neighborhood of any periodic point of \tilde{h} of period p contains uncountably many periodic points of \tilde{h} of period kp , for each $k \in \mathbb{N}$.
- (c) $R(\tilde{h}) = \Omega(\tilde{h})$ and $CR(\tilde{h}) = \mathcal{M}(\{0,1\}^{\mathbb{N}})$.
- (d) $\Omega(\tilde{f})$ has empty interior in $\mathcal{M}(\{0,1\}^{\mathbb{N}})$.
- (e) $P(\tilde{h})$ is dense in $\Omega(\tilde{h})$.

Proof. Let $h \in \mathcal{H}(\{0,1\}^{\mathbb{N}})$ satisfy property (P). We choose a partition \mathcal{P} of $\{0,1\}^{\mathbb{N}}$ such that every component of $\text{Gr}(h, \mathcal{P})$ is a balanced dumbbell with plate weight $q!$. Let

$$D_i := \{u_{i,1}, \dots, u_{i,q!}\} \cup \{v_{i,1}, \dots, v_{i,s_i}\} \cup \{w_{i,1}, \dots, w_{i,q!}\} \quad (1 \leq i \leq N)$$

be the components (dumbbells) of $\text{Gr}(h, \mathcal{P})$. We claim that, for every $\mu \in \Omega(\tilde{h})$,

$$\mu(v_{i,j}) = 0 \quad \text{and} \quad \mu(h^{-n}(v_{i,1})) = 0 \quad \text{for all } i, j \text{ and } n. \quad (21)$$

Indeed, let us fix $1 \leq i \leq N$ and $m \in \mathbb{N}$. It is enough to prove that $\mu(h^{-m}(v_{i,s_i})) = 0$. For this purpose, let b be the unique element of \mathcal{P} that contains $h^{-m}(v_{i,s_i})$. If b is a certain $v_{i,j}$, then $h^{-m}(v_{i,s_i}) = v_{i,j}$ and we define $\mathcal{P}' := \mathcal{P}$. Otherwise, we define \mathcal{P}' as the partition of $\{0,1\}^{\mathbb{N}}$ obtained from \mathcal{P} by replacing b by the sets $h^{-m}(v_{i,s_i})$ and $b \setminus h^{-m}(v_{i,s_i})$. Let $0 < \delta < \delta(\mathcal{P}')$. Since μ is a nonwandering point of \tilde{h} , there exist $t \in \mathbb{N}$ and $\nu \in \mathcal{M}(\{0,1\}^{\mathbb{N}})$ such that

$$d_P(\nu, \mu) < \frac{\delta}{2} \quad \text{and} \quad d_P(\tilde{h}^t(\nu), \mu) < \frac{\delta}{2}.$$

In particular, $d_P(\tilde{h}^t(\nu), \nu) < \delta$. By Lemma 2,

$$|\nu(h^{-nt}(a)) - \nu(a)| < n\delta \quad \text{for all } a \in \mathcal{P} \text{ and } n \in \mathbb{N}.$$

Let $k \in \mathbb{N}$ be the smallest positive integer such that $h^{-kt}(w_{i,j}) \supset w_{i,j}$ for all j . There are $1 \leq j \leq q!$ and $1 \leq n \leq m$ such that

$$h^{-nkt}(w_{i,j}) \supset h^{-m}(v_{i,s_i}).$$

Since $h^{-nkt}(w_{i,j}) \supset w_{i,j}$ and $|\nu(h^{-nkt}(w_{i,j})) - \nu(w_{i,j})| < nk\delta$, we conclude that

$$\nu(h^{-m}(v_{i,s_i})) < nk\delta \leq mq!\delta.$$

Since $d_P(\nu, \mu) < \delta < \delta(\mathcal{P}')$, Lemma 2 implies that

$$|\nu(h^{-m}(v_{i,s_i})) - \mu(h^{-m}(v_{i,s_i}))| < \delta.$$

Thus, $\mu(h^{-m}(v_{i,s_i})) < (mq! + 1)\delta$. Since $\delta > 0$ can be chosen arbitrarily small,

$$\mu(h^{-m}(v_{i,s_i})) = 0,$$

as was to be shown.

Let $\mu \in \Omega(\tilde{h})$. Given $\varepsilon > 0$, we may assume that \mathcal{P} was chosen with mesh $\mathcal{P} < \varepsilon$. It follows from (21) that

$$\mu(h^{-q!}(a)) = \mu(a) \quad \text{for all } a \in \mathcal{P}.$$

Hence, $d_P(\tilde{h}^{q!}(\mu), \mu) < \varepsilon$ by Lemma 3, which shows that $\mu \in R(\tilde{h})$. On the other hand, it follows immediately from Theorem 13 that $CR(\tilde{h}) = \mathcal{M}(M)$ for all $h \in \mathcal{H}(M)$. This proves property (c).

Let $\mu \in \mathcal{M}(\{0, 1\}^{\mathbb{N}})$ be arbitrary and choose a point $z \in v_{1,1}$. For each $\lambda \in (0, 1)$, define

$$\mu_\lambda := (1 - \lambda)\mu + \lambda\pi_z.$$

Then, $\mu_\lambda \notin \Omega(\tilde{h})$ (because of (21)) and $d_P(\mu_\lambda, \mu) \leq \lambda$ (by Lemma 7). This implies property (d).

Now, let $\mu \in R(\tilde{h})$ and $\varepsilon > 0$. We may assume that \mathcal{P} was chosen with mesh $\mathcal{P} < \varepsilon$. Fix a number δ satisfying $0 < \delta < \delta(\mathcal{P})$. Since μ is a recurrent point of \tilde{h} , there exists $p \in \mathbb{N}$ such that $d_P(\tilde{h}^p(\mu), \mu) < \delta$. Hence, it follows from Lemma 2 that

$$|\mu(h^{-np}(a)) - \mu(a)| < n\delta \quad \text{for all } a \in \mathcal{P} \text{ and } n \in \mathbb{N}. \quad (22)$$

As in the proof of Step 6 in Theorem 16, each “right loop” $\{w_{i,1}, \dots, w_{i,q!}\}$ can be partitioned in sets

$$\mathcal{A}_{i,r} := \{a_{i,r,0}, a_{i,r,1}, \dots, a_{i,r,k-1}\} \quad (1 \leq r \leq q!/k)$$

satisfying

$$a_{i,r,t} = h^{-tp}(a_{i,r,0}) \cap W_i \quad \text{for } 1 \leq t \leq k-1 \quad (23)$$

and

$$a_{i,r,0} = h^{-kp}(a_{i,r,0}) \cap W_i, \quad (24)$$

where $W_i := w_{i,1} \cup \dots \cup w_{i,q!}$ ($1 \leq i \leq N$). By (21), (22) and (23), μ is almost constant on $\mathcal{A}_{i,r}$. Define

$$d_{i,r} := \mu(a_{i,r,0}) + \mu(a_{i,r,1}) + \dots + \mu(a_{i,r,k-1}).$$

Then the average $d_{i,r}/k$ is very close to the values of μ on the elements of $\mathcal{A}_{i,r}$. More precisely, by choosing δ small enough, we can make the numbers

$$\left| \frac{d_{i,r}}{k} - \mu(a_{i,r,t}) \right|$$

as small as we want. Now, let us look at the “left loop” $\{u_{i,1}, \dots, u_{i,q!}\}$. Each of these “loops” can be partitioned in sets

$$\mathcal{B}_{i,r} := \{b_{i,r,0}, b_{i,r,1}, \dots, b_{i,r,k-1}\} \quad (1 \leq r \leq q!/k)$$

satisfying

$$h^{-tp}(b_{i,r,0}) \subset b_{i,r,t} \quad \text{for } 1 \leq t \leq k-1 \quad (25)$$

and

$$h^{-kp}(b_{i,r,0}) \subset b_{i,r,0}. \quad (26)$$

Note that

$$h^{-kp}(u_{i,j}) \subset u_{i,j} \quad (1 \leq i \leq N, 1 \leq j \leq q!).$$

Moreover, by (21),

$$\mu(u_{i,j} \setminus h^{-kp}(u_{i,j})) = 0.$$

This fact together with (22) and (25) imply that μ is almost constant on $\mathcal{B}_{i,r}$. Define

$$e_{i,r} := \mu(b_{i,r,0}) + \mu(b_{i,r,1}) + \dots + \mu(b_{i,r,k-1}).$$

Then the average $e_{i,r}/k$ is very close to the values of μ on the elements of $\mathcal{B}_{i,r}$. More precisely, by choosing δ small enough, we can make the numbers

$$\left| \frac{e_{i,r}}{k} - \mu(b_{i,r,t}) \right|$$

as small as we want. Now, by making a construction similar to the one in Step 1 of Theorem 16, we obtain uncountably many periodic points $\mu_{i,r}$ of \tilde{h} of period p such that

$$\mu_{i,r}(a_{i,r,0}) = \mu_{i,r}(a_{i,r,1}) = \dots = \mu_{i,r}(a_{i,r,k-1})$$

and

$$\mu_{i,r}(a) = 0 \quad \text{for all } a \in \mathcal{P} \setminus \{a_{i,r,0}, a_{i,r,1}, \dots, a_{i,r,k-1}\}.$$

Similarly, we can construct uncountably many periodic points $\nu_{i,r}$ of \tilde{h} of period p such that

$$\nu_{i,r}(b_{i,r,0}) = \nu_{i,r}(b_{i,r,1}) = \dots = \nu_{i,r}(b_{i,r,k-1})$$

and

$$\nu_{i,r}(b) = 0 \quad \text{for all } b \in \mathcal{P} \setminus \{b_{i,r,0}, b_{i,r,1}, \dots, b_{i,r,k-1}\}.$$

As in the proof of Step 6 of Theorem 16, we see that uncountably many measures of the form

$$\mu' := \sum_{i=1}^N \sum_{r=1}^{q!/k} (d_{i,r} \mu_{i,r} + e_{i,r} \nu_{i,r})$$

are periodic points of \tilde{h} of period p . Moreover, by choosing δ small enough, Lemma 3 implies that each of these measures satisfies $d_{\mathcal{P}}(\mu', \mu) < \varepsilon$. This establishes property (e). In the case μ is already a periodic point of \tilde{h} , of period t say, then we can choose p to be any multiple kt of t , and so we obtain uncountably many periodic points of \tilde{h} of period kt in the ε -neighborhood of μ . This gives property (b).

Finally, property (a) follows from the fact that a construction similar to the one in Step 1 of Theorem 16 can be made in the present context, as was already mentioned above. \square

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